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EQUATIONS OF MOTION OF SYSTEMS WITH SECOND-ORDER NONLINEAR NONHOLONOMIC CONSTRAINTS

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DUO SAN
(Hanoi)

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We derive the equations of motion of systems with second-order nonlinear nonholonomic constraints. As the foundation we choose the Gauss principle according to which the real accelerations ensure at each instant the minimum of the Gauss function in the class of all possible accelerations admissible by the system's constraints.

1. Various forms of the equations of motion of a system with nonlinear nonholonomic constraints. We consider a system of material points M_k with masses m_k $(k = 1, 2, \ldots, N)$, whose position is determined by the generalized coordinates q_i $(i = 1, 2, \ldots, n)$. We assume that second-order nonlinear nonholonomic constraints have been imposed on the system, i.e. constraints whose equations are [1]

$$f_\alpha (k, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}) = 0 \qquad (\alpha = 1, 2, \ldots, s;\ i = 1, 2, \ldots, n) \tag{1.1}$$

We apply the Gauss principle to derive the equations of motion of systems with constraints (1.1). To do this we set up the Gauss function

$$U = \frac{1}{2} \sum_{k=1}^{N} m_k (\mathbf{w}_k - \mathbf{F}_k / m_k)^2$$

Here $\mathbf{w}_k$ is the acceleration vector of material point M_k, $\mathbf{F}_k$ is a given force acting on the material point M_k. We expand the Gauss function

$$U = S - \sum_{k=1}^{N} \mathbf{F}_k \mathbf{w}_k + \frac{1}{2} \sum_{k=1}^{N} \mathbf{F}_k^2 / m_k, \qquad S = \frac{1}{2} \sum_{k=1}^{N} m_k \mathbf{w}_k^2 \tag{1.2}$$

The position of the material point M_k is determined by the vector $\mathbf{r}_k = \mathbf{r}_k (t, q_i)$, its velocity and acceleration are

$$\mathbf{v}_k = \mathbf{r}_k^{\cdot} = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_k}{\partial q_i} q_i^{\cdot} + \frac{\partial \mathbf{r}_k}{\partial t}$$

$$\mathbf{w}_k = \mathbf{r}_k^{\cdot\cdot} = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_k}{\partial q_i} q_i^{\cdot\cdot} + A$$

where A is a collection of terms not depending on $q_i^{\cdot\cdot}$. Consequently,

$$\mathbf{F}_k = \mathbf{F}_k (t, \mathbf{r}_k, \mathbf{v}_k) = \mathbf{F}_k (t, q_i, q_i^{\cdot}) \tag{1.3}$$

$$S = S (t, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}) \tag{1.4}$$

$$\partial \mathbf{w}_k / \partial q_i^{\cdot\cdot} = \partial \mathbf{r}_k / \partial q_i^{\cdot\cdot} \tag{1.5}$$

Using various methods of minimizing the Gauss function, we obtain various (equivalent) forms of the equations of motion.

1°. We compose the function

$$\Phi = U + \sum_{\alpha=1}^{s} \lambda_\alpha f_\alpha$$

Here λ_α are undetermined factors, f_α are constraint equations of the form of (1.1). The condition for the minimum of function U relative to the variables $\ddot{q}_i$ is $\partial\Phi/\partial\ddot{q}_i = 0$. Hence the equation of motion of the system is

$$\frac{\partial S}{\partial \ddot{q}_i} = Q_i - \sum_{\alpha=1}^{s} \lambda_\alpha \frac{\partial f_\alpha}{\partial \ddot{q}_i}, \qquad Q_i = \sum_{k=1}^{N} \mathbf{F}_k \frac{\partial \mathbf{r}_k}{\partial q_i} \tag{1.6}$$

$$(i = 1, 2, \ldots, n)$$

Here Q_i is the generalized force and S has the form (1.4). The system of constraint equations (1.1) should be added on to the Eqs. (1.6).

2°. We consider the case when the Jacobian

$$\frac{D(f_1, f_2, \ldots, f_s)}{D(\ddot{q}_{p+1}, \ddot{q}_{p+2}, \ldots, \ddot{q}_n)} \neq 0 \tag{1.7}$$

where $p = n - s$ is the number of independent generalized accelerations. We can then express the quantities $\ddot{q}_h$ in terms of $\ddot{q}_\nu$

$$\ddot{q}_h = \ddot{q}_h(t, q_i, \dot{q}_i, \ddot{q}_\nu), \qquad d^*\ddot{q}_h = \sum_{\nu=1}^{p} g_{h\nu} d^*\ddot{q}_\nu \tag{1.8}$$

$$(\nu = 1, 2, \ldots, p; \ h = p+1, \ p+2, \ldots, \ n)$$

Here $g_{h\nu} = g_{h\nu}(t, q_i, \dot{q}_i, \ddot{q}_\nu)$ and the differential operator d^* acts only on the variables $\ddot{q}_i$ (while the variables t, q_i, $\dot{q}_i$ should be taken to be constant).

From the condition of the minimum of the function U relative to the variables $\ddot{q}_i$ follows

$$\sum_{\nu=1}^{p} \left[\frac{\partial u}{\partial \ddot{q}_\nu} + \sum_{h=p+1}^{n} \frac{\partial U}{\partial \ddot{q}_h} g_{h\nu} \right] d^*\ddot{q}_\nu = 0$$

Taking (1.2) into account, we have the equations of motion in the form

$$\frac{\partial S}{\partial \ddot{q}_\nu} + \sum_{h=p+1}^{n} \frac{\partial S}{\partial \ddot{q}_h} g_{h\nu} = Q_\nu + \sum_{h=p+1}^{n} g_{h\nu} Q_h \qquad (\nu = 1, 2, \ldots, p) \tag{1.9}$$

$$Q_\nu = \sum_{k=1}^{N} \mathbf{F}_k \frac{\partial \mathbf{r}_k}{\partial q_\nu}, \qquad Q_h = \sum_{k=1}^{N} \mathbf{F}_k \frac{\partial \mathbf{r}_k}{\partial q_h} \tag{1.10}$$

Here S has the form (1.4).

3°. Having substituted (1.8) into function U, we obtain the equations of motion of the system being considered

$$\frac{\partial S}{\partial \ddot{q}_\nu} = Q_\nu + \sum_{h=p+1}^{n} g_{h\nu} Q_h \qquad (\nu = 1, 2, \ldots, p) \tag{1.11}$$

where $S = S(t, q_i, \dot{q}_i, \ddot{q}_\nu)$, while Q_ν and Q_h should be computed by formulas (1.10).

4°. Let

$$\ddot{\pi}_\nu = g_\nu(t, q_i, \dot{q}_i, \ddot{q}_i) \qquad (\nu = 1, 2, \ldots, p)$$

and let the condition

$$\frac{D(f_1, f_2, \ldots, f_s, g_1, g_2, \ldots, g_p)}{D(q_1^{\cdot\cdot}, q_2^{\cdot\cdot}, \ldots, q_n^{\cdot\cdot})} \neq 0 \qquad (1.12)$$

be fulfilled. Then the condition for the minimum of function U can be written in the form $\partial U / \partial \pi_\nu^{\cdot\cdot} = 0$. Taking (1.5) and the relation

$$\frac{\partial \mathbf{w}_k}{\partial \pi_\nu^{\cdot\cdot}} = \sum_{i=1}^{n} \frac{\partial \mathbf{w}_k}{\partial q_i^{\cdot\cdot}} \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}}$$

into account, we obtain the equations of motion of the system

$$\frac{\partial S}{\partial \pi_\nu^{\cdot\cdot}} = \sum_{i=1}^{n} Q_i \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}} = \Pi_\nu \qquad (\nu = 1, 2, \ldots, p) \qquad (1.13)$$

where $S = S(t, q_i, q_i^{\cdot}, \pi_\nu^{\cdot\cdot})$, while π_ν is the generalized force corresponding to the generalized pseudocoordinate.

Example. We consider the spherical motion of a rigid body with a nonholonomic constraint [1]. The condition of generalized precession of vector ω as a nonholonomic constraint has the form [1]

$$(pq^{\cdot} - dp^{\cdot}) + r(p^2 + q^2) - \lambda (p^2 + q^2)^{3/2} = 0$$

The body's acceleration energy has the form [1]

$$S = \frac{1}{2} [Ap^{\cdot 2} + Bq^{\cdot 2} + Cr^{\cdot 2} + 2(C - B) qrp^{\cdot} + 2(A - C) rpq^{\cdot} + 2(B - A) pqr^{\cdot} + \ldots]$$

Here and below the terms discarded do not contain $\psi^{\cdot\cdot}$, $\theta^{\cdot\cdot}$, $\varphi^{\cdot\cdot}$.

Using the Euler formulas we have

$$\begin{aligned} p^{\cdot} &= \psi^{\cdot\cdot} \sin\theta \sin\varphi + \theta^{\cdot\cdot} \cos\varphi + \ldots \\ q^{\cdot} &= \psi^{\cdot\cdot} \sin\theta \cos\varphi - \theta^{\cdot\cdot} \sin\varphi + \ldots \\ r^{\cdot} &= \psi^{\cdot\cdot} \cos\theta + \varphi^{\cdot\cdot} + \ldots \end{aligned}$$

We introduce the pseudo-accelerations $\pi_1^{\cdot\cdot}$, $\pi_2^{\cdot\cdot}$, by setting $\theta^{\cdot\cdot} = \theta^{\cdot}\pi_1^{\cdot\cdot}$, $\varphi^{\cdot\cdot} = \pi_2^{\cdot\cdot}$. Then

$$\psi^{\cdot\cdot} = \psi^{\cdot}\pi_1^{\cdot\cdot} + \ldots$$

$$p^{\cdot} = \pi_1^{\cdot\cdot} p + \ldots, \quad q^{\cdot} = \pi_1^{\cdot\cdot} q + \ldots, \quad r^{\cdot} = \pi_1^{\cdot\cdot}\psi^{\cdot} \cos\theta + \pi_2^{\cdot\cdot} + \ldots$$

Terms not containing $\pi_1^{\cdot\cdot}$, $\pi_2^{\cdot\cdot}$ have been discarded in the right-hand sides of these formulas.

Noting that

$$\frac{\partial S}{\partial \pi_j^{\cdot\cdot}} = \frac{\partial S}{\partial p^{\cdot}} \frac{\partial p^{\cdot}}{\partial \pi_j^{\cdot\cdot}} + \frac{\partial S}{\partial q^{\cdot}} \frac{\partial q^{\cdot}}{\partial \pi_j^{\cdot\cdot}} + \frac{\partial S}{\partial r^{\cdot}} \frac{\partial r^{\cdot}}{\partial \pi_j^{\cdot\cdot}} \qquad (j = 1, 2)$$

we obtain the equations of motion

$$Cr^{\cdot} - (A - B) pq = Q_\varphi$$

$$Cr^{\cdot}\psi^{\cdot} \cos\theta + App^{\cdot} + Bqq^{\cdot} + (A - B) pq\varphi^{\cdot} = \psi^{\cdot} Q_\psi + \theta^{\cdot} Q_\theta$$

Thus, we arrive by a simpler path to the result which follows from Tzénoff's equations [1].

5°. Using the identities

$$\frac{\partial S}{\partial q_i^{\cdot\cdot}} = \frac{d}{dt} \frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} \qquad (1.14)$$

where $T = T(t, q_i, q_i^{\cdot})$ is the system's kinetic energy, and S has the form (1.4), we write Eq. (1.6) as

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = Q_i - \sum_{\alpha=1}^{s} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i^{\cdot\cdot}} \quad (i = 1,2, \ldots, n) \tag{1.15}$$

6°. Taking relations (1.14) into account, Eqs. (1.9) can be represented in the form

$$\frac{d}{dt}\frac{\partial T}{\partial q_\nu^{\cdot}} - \frac{\partial T}{\partial q_\nu} + \sum_{n=p+1}^{n} g_{h\nu}\left(\frac{d}{dt}\frac{\partial T}{\partial q_h^{\cdot}} - \frac{\partial T}{\partial q_h}\right) =$$

$$Q_\nu + \sum_{h=p+1}^{n} g_{h\nu} Q_h \qquad (\nu = 1,2, \ldots, p) \tag{1.16}$$

Here $T = T(t, q_i, q_i^{\cdot})$.

Appel's example (see [2, 3]). Taking into account the remark in [3], we can write the expressions for the kinetic and potential energies and the constraint equation as

$$T = {}^1/_2 m (x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}), \qquad \Pi = -mgz$$

$$z^{\cdot 2} - a^2 (x^{\cdot 2} + y^{\cdot 2}) = 0$$

Hence

$$f(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, x^{\cdot\cdot}, y^{\cdot\cdot}, z^{\cdot\cdot}, t) = z^{\cdot} z^{\cdot\cdot} - a^2 (x^{\cdot} x^{\cdot\cdot} + y^{\cdot} y^{\cdot\cdot}) = 0$$

After simple manipulations, from Eqs. (1.16) we have

$$x^{\cdot\cdot} + \frac{a^2 x^{\cdot} (x^{\cdot} x^{\cdot\cdot} + y^{\cdot} y^{\cdot\cdot})}{x^{\cdot 2} + y^{\cdot 2}} = -ga \frac{x^{\cdot}}{\sqrt{x^{\cdot 2} + y^{\cdot 2}}}$$

$$y^{\cdot\cdot} + \frac{a^2 y^{\cdot} (x^{\cdot} x^{\cdot\cdot} + y^{\cdot} y^{\cdot\cdot})}{x^{\cdot 2} + y^{\cdot 2}} = -ga \frac{y^{\cdot}}{\sqrt{x^{\cdot 2} + y^{2}}}$$

The equations obtained agree with those established in [3].

7°. We use the relations

$$\frac{\partial S}{\partial \pi_\nu^{\cdot\cdot}} = \sum_{i=1}^{n} \frac{\partial S}{\partial q_i^{\cdot\cdot}} \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}} = \sum_{i=1}^{n} \left(\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i}\right) \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}}$$

Then from (1.13) we have

$$\sum_{i=1}^{n} \left(\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i}\right) \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}} = \sum_{i=1}^{n} Q_i \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}} \qquad (\nu = 1,2, \ldots, p) \tag{1.17}$$

Here

$$T = T(t, q_i, q_i^{\cdot})$$

Example. We examine the motion of a material point in a central field of Newtonian attraction forces. The absolute value of the point's velocity is a constant. By considering the motion in the spherical coordinates r, φ, θ with origin at the attracting center, we represent the constraint in the form [1]

$$f(r, \theta, \varphi, r^{\cdot}, \theta^{\cdot}, \varphi^{\cdot}) = \frac{1}{2} m (r^{\cdot 2} + r^2 \cos^2\theta \varphi^{\cdot 2} + r^2 \theta^{\cdot 2}) = \text{const} \tag{1.18}$$

This equation can be written as

$$r^{\cdot} r^{\cdot\cdot} + r^2 \cos^2\theta \varphi^{\cdot} \varphi^{\cdot\cdot} + r^2 \theta^{\cdot} \theta^{\cdot\cdot} + \ldots = 0$$

where the terms not written down do not contain $r^{\cdot\cdot}$, $\theta^{\cdot\cdot}$, $\varphi^{\cdot\cdot}$.

We introduce the pseudo-accelerations $\pi_1^{\cdot\cdot}$, $\pi_2^{\cdot\cdot}$, by setting $\pi_1^{\cdot\cdot} = r^2 \theta^{\cdot} \theta^{\cdot\cdot}$, $\pi_2^{\cdot\cdot} = r^{\cdot} r^{\cdot\cdot}$. Then

$$\varphi^{\cdot\cdot} = \frac{1}{r^2 \cos^2 \theta \varphi^{\cdot}} (\pi_1^{\cdot\cdot} + \pi_2^{\cdot\cdot}) + \ldots$$

where the discarded terms do not contain $\pi_1^{\cdot\cdot}$, $\pi_2^{\cdot\cdot}$. After simple manipulations, from Eqs. (1.17) we have

$$r^{\cdot\cdot} - r\theta^{\cdot 2} - \frac{r^{\cdot}}{\varphi^{\cdot}} \varphi^{\cdot\cdot} - 2\frac{r^{\cdot 2}}{r} + 2\theta^{\cdot} r^{\cdot} \operatorname{tg}\theta - 2\varphi^{\cdot 2} \cos^2\theta = -\gamma \frac{M}{r^2}$$

$$r\theta^{\cdot\cdot} + r^2\varphi^{\cdot 2} \sin\theta\cos\theta - r^2 \frac{\theta^{\cdot}}{\varphi^{\cdot}} \varphi^{\cdot\cdot} + 2r^2\theta^{\cdot 2} \operatorname{tg}\theta = 0$$

These equations together with the constraint equation (1.18) allow us to find the unknowns desired. The equations obtained have the very same form as in [3] if in the latter we eliminate the factors.

Note. The equations written down are obviously applicable also for systems with holonomic constraints, linear nonholonomic constraints, and first-order nonlinear nonholonomic constraints. Equations (1.16), (1.17) simplify the computations for setting up the equations of motion. If the generalized velocities enter linearly in the constraint equations, then these equations agree with the Magie equation [1, 2].

2. The Nielsen equation for systems with second-order nonlinear nonholonomic constraints. Using the relations

$$T^{\cdot} = \sum_{k=1}^{N} m_k \mathbf{v}_k \mathbf{w}_k, \qquad \frac{\partial \mathbf{w}_k}{\partial q_i^{\cdot}} = \frac{\partial \mathbf{v}_k}{\partial q_i}, \qquad \frac{\partial \mathbf{w}_k}{\partial q_i^{\cdot\cdot}} = \frac{\partial \mathbf{v}_k}{\partial q_i^{\cdot}}$$

it is easy to prove the identity

$$\frac{\partial S}{\partial q_i^{\cdot\cdot}} = \frac{\partial T^{\cdot}}{\partial q_i^{\cdot}} - 2\frac{\partial T}{\partial q_i}, \quad T = T(t, q_i, q_i^{\cdot}), \quad T^{\cdot} = dT/dt \tag{2.1}$$

Here S is of form (1.4). The system's equations of motion take one of the following forms:

$$\frac{\partial T^{\cdot}}{\partial q_i^{\cdot}} - 2\frac{\partial T}{\partial q_i} = Q_i - \sum_{\alpha=1}^{s} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i^{\cdot\cdot}}$$

$$\frac{\partial T^{\cdot}}{\partial q_\nu^{\cdot}} - 2\frac{\partial T}{\partial q_\nu} + \sum_{h=p+1}^{n} g_{h\nu} \left(\frac{\partial T^{\cdot}}{\partial q_h^{\cdot}} - 2\frac{\partial T}{\partial q_h}\right) = Q_\nu + \sum_{h=p+1}^{n} Q_h g_{h\nu} \tag{2.2}$$

$$\sum_{i=1}^{n} \left(\frac{\partial T^{\cdot}}{\partial q_i^{\cdot}} - 2\frac{\partial T}{\partial q_i}\right) \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}} = \sum_{i=1}^{n} Q_i \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}}$$

Let us consider the motion of a system with the first-order nonholonomic constraints

$$\Phi_\alpha (t, q_i, q_i^{\cdot}) = 0 \tag{2.3}$$

Equations (2.3) can be written as

$$f_\alpha (t, q_i, q_i^{\cdot}, q_i^{\cdot\cdot}) = \sum_{i=1}^{n} \frac{\partial \Phi_\alpha}{\partial q_i^{\cdot}} q_i^{\cdot\cdot} + \ldots = 0 \tag{2.4}$$

where the terms not written out do not contain $q_i^{\cdot\cdot}$. From (2.4) follows the relation $\partial f_\alpha / \partial q_i^{\cdot\cdot} = \partial \Phi_\alpha / \partial q_i^{\cdot}$ and, consequently, the first equation in (2.2) becomes

$$\frac{\partial T^{\cdot}}{\partial q_i^{\cdot}} - 2\frac{\partial T}{\partial q_i} = Q_i - \sum_{\alpha=1} \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial q_i^{\cdot}} \tag{2.5}$$

Finally, if holonomic constraints are imposed on the system, then $\lambda_\alpha \equiv 0$ in (2.5).

Equations (2.5) obtained agree with those presented in [1].

3. The Tzénoff's equation of the second kind. It can be shown that

$$T^{\cdot\cdot} = 2S + 3\sum_{k=1}^{N} m_k \mathbf{v}_k \mathbf{w}_k^{\cdot} + B \tag{3.1}$$

where B is a collection of terms not containing the first-order derivatives of the accelerations of the points. Since

$$\frac{\partial \mathbf{w}_k^{\cdot}}{\partial q_i^{\cdot\cdot}} = \frac{\partial \mathbf{v}_k}{\partial q_i}, \qquad \mathbf{w}_k^{\cdot} = \sum_{i=1}^{n} \frac{\partial \mathbf{v}_k}{\partial q_i} q_i^{\cdot\cdot} + C$$

from (3.1) we obtain

$$S = p + D, \qquad p = \frac{1}{2}\left[T^{\cdot\cdot} - 3\sum_{i=1}^{n} \frac{\partial T}{\partial q_i} q_i^{\cdot\cdot}\right]$$

Here C, D are collections of terms not containing $q_i^{\cdot\cdot}$. Now the system's equations of motion take one of the following three forms:

$$\frac{\partial R}{\partial q_i^{\cdot\cdot}} - \sum_{\alpha=1}^{s} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i^{\cdot\cdot}} = 0, \qquad \sum_{i=1}^{n} \frac{\partial R}{\partial q_\nu^{\cdot\cdot}} \frac{\partial q_i^{\cdot\cdot}}{\partial \pi_\nu^{\cdot\cdot}} = 0$$

$$\frac{\partial R}{\partial q_\nu^{\cdot\cdot}} + \sum_{h=p+1}^{n} \frac{\partial R}{\partial q_h^{\cdot\cdot}} g_{h\nu} = 0 \qquad \left(R = p - \sum_{i=1}^{n} Q_i q_i^{\cdot\cdot}\right)$$

The last equation agrees with the one presented in [1].

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